From thin to thick granular surface flows: The stop flow problem

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The surface evolution of a sandpile was investigated theoretically by Bouchaud *et al.* several years ago [J. Phys. I **4**, 1383 (1994)]. Their model assumes that the erosion/accretion rate of the static grains is proportional to the local amount R of rolling species. de Gennes *et al.* [Phys. Rev. E **58**, 4692 (1998)] have noticed recently that this assumption must be modified for thick surface flows, where the rate should become less dependent on R. In order to analyze the progressive transition from thin to thick flows, we focus on the so-called *stop flow* problem, where an incoming front of rolling particles suddenly hits an immobile wall. We find that the physical behavior of the grains changes significantly as the thickness of the incoming front increases. [S1063-651X(98)08412-8]

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I. SURFACE FLOWS OF VARIOUS THICKNESSES

Several years ago, Bouchaud, Cates, Ravi Prakash, and Edwards introduced a model to describe surface flows of ganular matter [1,2]. The model assumes a rather sharp distinction between *immobile* particles and *rolling* particles and, accordingly, introduces the following two important physical quantities (see Fig. 1): the local height of immobile particles $h(x,t)$ (where x denotes the horizontal coordinate and t the time) and the local amount of rolling particles $R(x,t)$. The time evolution of $h(x,t)$ is written in the form

$$\frac{\partial h}{\partial t}=\gamma R(\theta_n-\theta), \tag{1}$$

where $\theta\simeq\tan(\theta)=\partial h/\partial x$ is the local slope, γ a characteristic frequency, and θ_n the neutral angle of grains at which erosion of the immobile grains balances accretion of the rolling grains. For the rolling particles, Bouchaud, Cates, Ravi Prakash, and Edwards wrote a convective-diffusive equation [2] that was later simplified by de Gennes as [3]

$$\frac{\partial R}{\partial t}=v\ \frac{\partial R}{\partial x}-\frac{\partial h}{\partial t}, \tag{2}$$

where v is the downhill convection velocity of the rolling grains, assumed to be approximately constant. Dimensionally, one expects v to scale like $(gd)^{1/2}$ (where g is the gravity acceleration and d the particle diameter) and γ to scale like v/d [3]. According to the Bouchaud–Cates–Ravi Prakash–Edwards (BCRE) model, $\partial h/\partial t$ is linear in R [see Eq. (1)]. This is natural at small R, when all the rolling grains interact with the immobile particles. However, as explained in Refs. [4,5], this cannot hold when R becomes lager than a given *saturation length* ξ' since the grains in the upper part of the rolling phase are no longer in contact with the immobile grains. The length ξ' is expected to be of the order of a few grain diameters d [6]. This led Boutreux, Raphaël, and de Gennes to propose [5] a modified version of the BCRE equation (1), valid for thick surface flows and of the form

$$\frac{\partial h}{\partial t}=v_{\mathrm{up}}(\theta_n-\theta)\quad (R\gg\xi'), \tag{3}$$

where v_{up} is defined by

$$v_{\mathrm{up}}\equiv\gamma\xi'. \tag{4}$$

The constant v_{up} has the dimensions of a velocity and has an order of magnitude comparable to v. The description of thick avalanches modelized by Eq. (3) was discussed in Ref. [5].

In some situations, the local amount R of rolling particles is large [and thus Eq. (3) is valid] except in some regions of space where R takes values smaller than ξ'. For example, in the case of thick avalanches in a closed cell [5], R has to vanish near the stopping wall. In some other situations, R is everywhere comparable to ξ' and neither Eq. (1) ($R<\xi'$) nor Eq. (3) ($\xi'\ll R$) gives a good description of the time

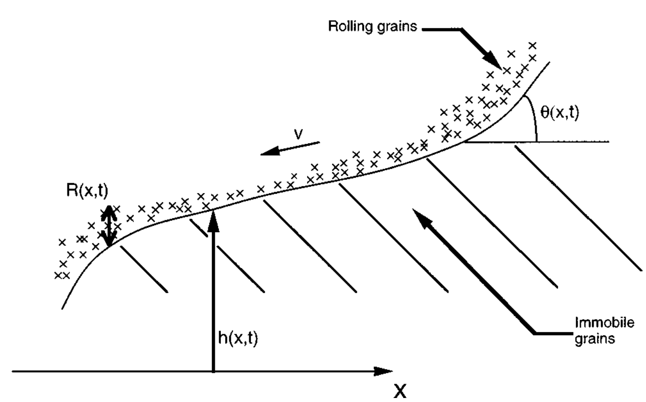

FIG. 1. The basic assumption of the BCRE picture is that there is a sharp distinction between immobile grains with a profile $h(x,t)$ and rolling particles with a local amount $R(x,t)$. The local slope of the static grains is called $\theta(x,t)$.

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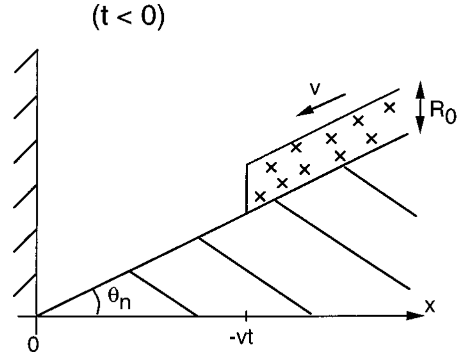

FIG. 2. Schematic of the *stop flow* problem for $t<0$. The immobile particles are described by a profile $h=\theta_n x$, limited by a stopping wall located at $x=0$. The rolling particles are convected downwards and are described by $R=R_0\mathcal{S}(x+vt)$, where $\mathcal{S}(u)=1$ for $u>0$ and $\mathcal{S}(u)=0$ for $u<0$.

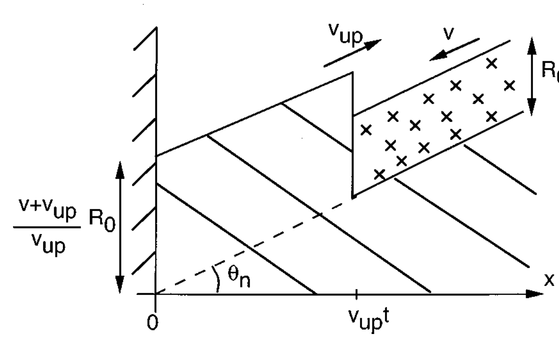

FIG. 3. Profiles for $t>0$ in the case of a thick incoming front ($R_0 \gg \xi'$). All the rolling grains stop when they meet a kink that propagates uphill with a speed v_{up}.

evolution of $h(x,t)$. It is therefore important to have an equation for $\partial h/\partial t$ valid for all values of R. We may interpolate between the two limits $R<\xi'$ and $\xi' \ll R$ as

$$\frac{\partial h}{\partial t}=\gamma \xi' F\left(\frac{R}{\xi'}\right)(\theta_n-\theta), \tag{5}$$

where the unknown function $F(x)$ has the limiting behaviors

$$F(x\rightarrow 0)\simeq x, \tag{6}$$

$$F(x\gg 1)\simeq 1. \tag{7}$$

If one makes the simple choice [4]

$$F(x)=\frac{x}{1+x}, \tag{8}$$

Eq. (5) can be rewritten as [4]

$$\frac{\partial h}{\partial t}=\gamma \frac{R\xi'}{R+\xi'}(\theta_n-\theta). \tag{9}$$

Equation (9) is valid for any value of R. In the present paper we will compare the predictions of Eqs. (1), (3), and (9) for a particular surface flow for which these three equations can be solved analytically.

II. COMPARISON BETWEEN THICK AND THIN SURFACE FLOWS

A. Stop flow problem

Let us focus on a particular situation recently introduced by de Gennes, the so-called *stop flow* problem [7]. The situation is described in Fig. 2. We consider a pile of immobile particles in the region $x>0$, limited by a *wall* at $x=0$. The pile is exactly at the neutral angle θ_n,

$$h=\theta_n x. \tag{10}$$

By a suitable feeding device, we add atop of the pile a *front* of rolling particles. At time $t<0$, the front is simply convected downhill and is described by

$$R=R_0\mathcal{S}(x+vt) \quad (t<0), \tag{11}$$

where $\mathcal{S}(u)=1$ for $u>0$ and $\mathcal{S}(u)=0$ for $u<0$. This initial situation is possible since Eq. (10) is a solution of any of the evolution equations (1), (3), or (9) ($\partial h/\partial t=0$ and $\theta=\partial h/\partial x=\theta_n$) and Eq. (11) is a of solution of Eq. (2).

The incoming front located at $x=-vt$ hits the blocking wall at the time $t=0$. For $t>0$, the grains pile up at the bottom of the slope since the wall imposes the boundary condition $R(x=0,t)=0$. We are now going to compare three different descriptions of the way the rolling grains stop and pile up, using successively Eqs. (3), (1), and (9).

B. Thick fronts

Let us first consider the case of a *thick* incoming front (i.e., $R_0 \gg \xi'$) and assume that the two quantities $h(x,t)$ and $R(x,t)$ satisfy Eqs. (2) and (3). The general solution of these equations was analytically calculated in [5] and is given by

$$h(x,t)=\theta_n x+g(x-v_{\mathrm{up}}t), \tag{12}$$

$$R(x,t)=f(x+vt)-\frac{v_{\mathrm{up}}}{v+v_{\mathrm{up}}}g(x-v_{\mathrm{up}}t), \tag{13}$$

where f and g are two arbitrary functions that for a given situation are determined by the initial and the boundary conditions. The function f describes a downward convection at the velocity v, related to the motion of the rolling grains. The function g describes an upward convection at the constant speed v_{up}; it corresponds to the *uphill waves* for the profile of the static phase described by Bouchaud *et al.* [2].

Let us determine the two functions $f(z)$ and $g(z)$ for the stop flow problem. The initial conditions $h(x>0,t=0)=\theta_n x$ and $R(x>0,t=0)=R_0$ yield $g(z>0)=0$ and $f(z>0)=R_0$. The boundary condition $R(x=0,t)=0$ and Eq. (13) imply that we have

$$g(-v_{\mathrm{up}}t)=\frac{v+v_{\mathrm{up}}}{v_{\mathrm{up}}}f(vt). \tag{14}$$

This leads to the expression of g for $z<0$: $g(z<0)=R_0(v+v_{\mathrm{up}})/v_{\mathrm{up}}$. Knowing f and g, we can now use Eqs. (12) and (13) to obtain h and R. The pile has to be divided into two regions. In the lower part of the pile ($0<x<v_{\mathrm{up}}t$), we have a *perturbed region* where there are no more rolling grains,

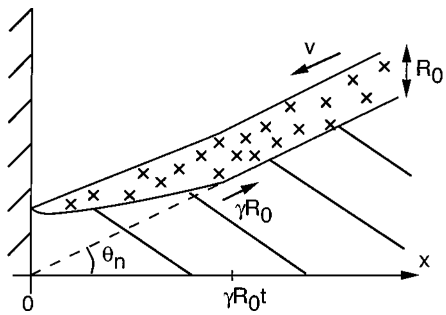

FIG. 4. Profiles for $t>0$ in the case of a thin incoming front ($R_0<\xi'$). The rolling grains progressively stop in the perturbed region $0<x<\gamma R_0 t$, where the slope of the pile is nearly constant and smaller than the neutral angle θ_n.

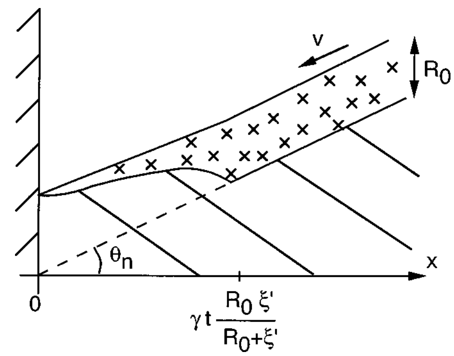

FIG. 5. Profiles for $t>0$ in the case of an intermediate incoming front (R_0 comparable to ξ'). The stopping zone where the rolling phase ends has a finite width and is mostly located to the left of the point $x=\gamma t R_0 \xi'/(R_0+\xi')$.

$$R(x,t)=0, \tag{15}$$

$$h(x,t)=\theta_n x+\frac{v+v_{\mathrm{up}}}{v_{\mathrm{up}}} R_0 . \tag{16}$$

For $x>v_{\mathrm{up}}t$, we have an *unperturbed region* where the effect of the wall is not yet experienced,

$$R(x,t)=R_0, \tag{17}$$

$$h(x,t)=\theta_n x. \tag{18}$$

This region is unperturbed because it has not been reached by the uphill waves, generated when the front hits the wall at $t=0$ and located at $x=v_{\mathrm{up}}t$ at time t. The profiles of R and h at a given time are represented in Fig. 3. To summarize, the solution of the stop flow problem for a thick incoming front shows a kink located at $x=v_{\mathrm{up}}t$ and propagating upward. All the rolling grains stop when they meet the kink, where both $R(x,t)$ and $h(x,t)$ are discontinuous. The final slope behind the kink is still equal to θ_n. Note that in the region where rolling grains are present, the thickness of the flow is larger that ξ', thus justifying the use of Eq. (9).

C. Thin fronts

Let us now consider the stop flow problem in the case of a *thin* incoming front, i.e., $R_0<\xi'$. The two quantities $h(x,t)$ and $R(x,t)$ must then satisfy the usual BCRE equations (1) and (2). This situation was already considered by de Gennes [7].

Again, we expect to find at the bottom of the pile a perturbed region already invaded by the uphill waves and an unperburbed region for large values of x. The uphill waves now propagate with the local velocity $\gamma R(x,t)$. Since this velocity increases with the thickness R of the rolling particles, no kink in the profiles R and h can be present [8] (i.e., R and h are continuous functions). Hence, at the boundary between the two regions, we expect the thickness of the rolling front to be equal to R_0; the local velocity of the waves at the boundary is then equal to γR_0. At time t, the perturbed region spreads between $x=0$ and $x_{\mathrm{up}}(t)\equiv\gamma R_0 t$. The region $x>x_{\mathrm{up}}(t)$ is still unperturbed and satisfies $R=R_0$ and $h=\theta_n x$. In the perturbed region, we assume that the physical quantities depend only on the dimensionless variable $u\equiv x/x_{\mathrm{up}}(t)$. We will use the boundary conditions $R(u=0)=0$, $R(u=1)=R_0$, and $\tilde{h}(u=1)=0$, where the reduced profile $\tilde{h}$ is defined by $\tilde{h}=h-\theta_n x$. Using Eqs. (1) and (2) along with the above boundary conditions, we obtain the exact solution for $u\leqslant 1$,

$$R(u)=R_0 u, \tag{19}$$

$$\tilde{h}(u)=R_0(1-u)-\frac{v}{\gamma}\ln(u). \tag{20}$$

Returning to the variables x and t, we get the expressions of the solution in the perturbed region $x<\gamma R_0 t$,

$$R(x,t)=R_0 \frac{x}{\gamma R_0 t}, \tag{21}$$

$$h(x,t)=\theta_n x+R_0\left(1-\frac{x}{\gamma R_0 t}\right)-\frac{v}{\gamma}\ln\left(\frac{x}{\gamma R_0 t}\right). \tag{22}$$

Equation (21) shows that for a thin incoming front, the rolling particles are present in the whole perturbed region $0\leqslant x\leqslant\gamma R_0 t$ and that their amount decreases linearly when one approches the wall (see Fig. 4). The local slope is equal to $\partial h/\partial x=\theta_n-1/\gamma t-v/\gamma x$. Since the last term is very small (of the order of d/x, where d is the grain size), the profile of $h(x,t)$ is nearly linear. The slope is smaller than the neutral angle θ_n of the grains. As soon as $t\gg\gamma^{-1}\sim 10^{-2}$ s, the slope is close to θ_n, except at the very bottom of the pile ($x\sim d$).

D. Comparison between the two previous descriptions

We have presented two rather different descriptions of the stop flow problem: one valid when we have a thick incoming front ($R_0\gg\xi'$) and the other valid for a thin incoming front ($R_0<\xi'$). In the first description, the stopping of the rolling particles occurs at a point ($x=v_{\mathrm{up}}t$), while in the second description the stopping occurs over a large region ($0\leqslant x\leqslant\gamma R_0 t$). Qualitatively, the origin of this difference can be understood as follows. In the case of a thin front, all the rolling particles interact with the immobile phase and progressively come to rest since the slope of the pile is smaller than θ_n. For a thick front, the particles located in the upper part of the rolling phase do not interact with the immobile phase (since $R_0 \gg \xi'$). These particles stop brutally as they hit the wall and jamming occurs as further particles arrive. This leads to the formation of a kink that moves backward with speed v_{up}.

III. INTERPOLATION MODEL

In order to quantitatively link the previous two descriptions, we are now going to solve the stop flow problem in the general case of an incoming front of arbitrary thickness R_0. The surface flows is then described by Eqs. (2) and (9) and $\partial h/\partial t$ is proportional to $R\xi'/(R+\xi')$. As in the two previous cases, we expect to find at time t a perturbed region in between the two points $x=0$ and $x_{\mathrm{up}}(t)$, where $x_{\mathrm{up}}(t)$ is now defined by

$$x_{\mathrm{up}}(t)=\gamma t\,\frac{R_0\xi'}{R_0+\xi'}. \tag{23}$$

As with a thin incoming front, we look for a solution to Eqs. (2) and (9) that depends only on the reduced variable $u \equiv x/x_{\mathrm{up}}$; this solution must satisfy the boundary condition $R(u=0)=0$ and must also cross over with the solution for the unperturbed region: $R(u=1)=R_0$ and $\tilde{h}(u=1)=0$. We thus obtain for the perturbed region $[0<x<x_{\mathrm{up}}(t)]$ the solution

$$R(x,t)=\frac{\xi'}{\xi'\gamma t/x-1}, \tag{24}$$

$$h(x,t)=\theta_n x+\left(1+\frac{v}{\gamma\xi'}\right)\left(R_0-\frac{\xi'}{\xi'\gamma t/x-1}\right)-\frac{v}{\gamma}\ln\!\left(\frac{\xi'}{R_0}\,\frac{1}{\xi'\gamma t/x-1}\right). \tag{25}$$

As before, we have an unperturbed region for $x>x_{\mathrm{up}}(t)$, where $R=R_0$ and $h=\theta_n x$. In order to show that this new solution (24) and (25) properly interpolates between the two limiting cases that we have already discussed, we now estimate the width of the stopping zone described by Eqs. (24) and (25). Let us call x_h the point for which R is equal to $R_0/2$. Using Eq. (24) we obtain $x_h=\gamma t R_0\xi'/(R_0+2\xi')$. If we define the *relative width* δ of the stopping zone by $\delta \equiv 2(x_{\mathrm{up}}-x_h)/x_{\mathrm{up}}$ we get

$$\delta=\frac{1}{1+\frac{1}{2}R_0/\xi'}. \tag{26}$$

Equation (26) indicates that the ratio R_0/ξ' is a key parameter to describe the stopping zone.

(i) If $R_0\gg\xi'$, the relative width $\delta\simeq 2\xi'/R_0$ is much smaller than one, i.e., $x_h\sim x_{\mathrm{up}}$, and we recover a "kink." In the region $0\leq x<x_{\mathrm{up}}\simeq v_{\mathrm{up}}t$, the amount of rolling particles is very small, R being of the order of ξ' ($\ll R_0$). These results are similar to those found for a thick incoming front using Eq. (3) (see Fig. 3).

(ii) If $R_0\ll\xi'$, we obtain $\delta\simeq 1$, i.e., $x_h\simeq x_{\mathrm{up}}/2$: The stopping of the rolling particles occurs over the whole region $0\leq x\leq x_{\mathrm{up}}\simeq\gamma R_0 t$ and the amount of rolling particles decreases linearly when one approaches the wall. In this case, we recover the results found for a thin incoming front using Eq. (1) (see Fig. 4).

(iii) If R_0 is comparable to ξ', the relative width δ is finite but smaller than 1. This situation, represented in Fig. 5, is intermediate between the two previous cases: (a) A large part of the rolling grains stops just below the point $x_{\mathrm{up}}=\gamma t R_0\xi'/(R_0+\xi')$, where we have a small blunt kink, and (b) at the same time, the whole perturbed region still contains a finite fraction of the incoming rolling particles.

IV. CONCLUDING REMARKS

To summarize, in the present paper we have described how an immobile wall stops a front of rolling grains. The first two physical situations that we have modeled correspond respectively to fronts of amplitude R_0 much thicker and much thinner than the saturation length ξ'. The results we found for these two situations are rather different. In particular, the width and the shape of the stopping zone vary significantly from one situation to the other. These differences stress that thick surface flows of granular matter should not be described with the usual BCRE equations, valid only for thin flows. Very recently, Sauermann and Herrmann have also noticed the need of saturation effects to describe thick flows [9]. The two different behaviors, expected for thin and thick flows in the stop flow problem, could in principle be observed in experiments in cells [10] or numerical simulations.

We have also proposed a model that interpolates between thick and thin flows and solved this model analytically for the stop flow problem. For a thick surface flow, the linear equation (3) leads to the formation of a kink where R and h are discontinuous. We have seen that a more realistic picture (with no discontinuity) can be obtained by using the full equation (9). Far away from the "kink zone," Eqs. (3) and (9) lead, however, to similar results. This shows that one can use the linear equation (3) to describe situations where the local amount R of rolling particles is large except in some small regions of space, where R takes values smaller than ξ' (e.g., near a kink or a stopping wall). This gives support to our previous model [based on Eq. (3)] of the dynamics of thick avalanches in a cell [5].

Our model has some limitations. (i) In Secs. II C and III we searched for solutions depending only on the reduced variable $u\propto x/t$. While this choice seems natural on physical grounds, other solutions might exist. (ii) A granular material is always, to some extent, polydisperse. This may have dramatic effects on surface flows. Boutreux and de Gennes [11] and Makse and co-workers [12] have recently extended the BCRE equations for a mixture of two types of grains. We are currently working on the extension of our model [Eqs. (2) and (9)] for the case of a mixture.

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